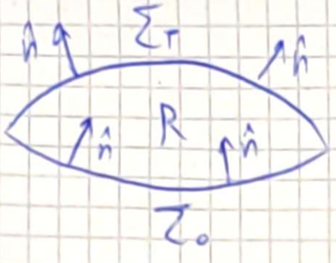
# General Relativity | Week 11




Let us consider the initial value problem

$$\begin{cases} \Box_g \psi = F & \text{on } R \\ \psi|_{\Sigma_0} = \psi_0, \quad \hat{n}(\psi)|_{\Sigma_0} = \psi_1 \end{cases} \qquad ①$$

for $\psi_0, \psi_1 \in C^\infty(\Sigma_0)$, $F \in C^\infty(R)$. We want to show the <u>existence</u> of solutions

$\rightsquigarrow$ It suffices to show it in the case when $\psi_0 = 0$, $\psi_1 = 0$.

In this case: <u>Weak</u> (or <u>distributional</u>) formulation of ①:

If $X = \{ \phi \in C^\infty(R) : \phi|_{\Sigma_T} = 0, \ \hat{n}(\phi)|_{\Sigma_T} = 0 \}$ (space of test functions)

Then: if $\psi$ solves ① with $(\psi_0, \psi_1) = 0$, then $\forall \phi \in X$:

$$\int_R F \cdot \phi = \int_R \Box_g \psi \cdot \phi \overset{\text{divergence thm}}{=} - \int_{\Sigma_T} \cancel{\left( \hat{n}(\psi) \cdot \phi - \psi \cdot \hat{n}(\phi) \right)}_{0} + \int_{\Sigma_0} \cancel{\left( \hat{n}(\psi) \phi - \psi \cdot \hat{n}(\phi) \right)}_{0}$$
$$+ \int_R \psi \cdot \Box_g \phi \quad \Rightarrow$$

$$\Rightarrow \boxed{\int_R F \cdot \phi = \int_R \psi \cdot \Box_g \phi \quad \forall \phi \in X} \qquad ②$$

(equivalent to ① if $\psi \in C^2$ by integrating by parts again)

For our <u>first proof of existence</u>: We will show the existence of a $\psi$ satisfying ②.

~~Consider the functional $L : X \to \mathbb{R}$, $L[\phi]$~~

Set $Y = \{ h \in C^\infty(R) : h = \Box_g \phi \text{ for some } \phi \in X \}$

and define

$$L : Y \to \mathbb{R} \quad \text{by} \quad L[\Box_g \phi] = \int_R F \cdot \phi.$$

Then $|L[\Box_g \phi]| \le \int_R |F| \cdot |\phi| \le \|F\|_{L^2(R)} \, \|\phi\|_{L^2(R)}$.

But since $\phi$ has zero initial data on $\Sigma_T$: Applying the energy identity on the trivial initial value problem $\begin{cases} \Box_g \phi = \Box_g \phi \\ \phi|_{\Sigma_T} = 0, \ \hat{n}(\phi)|_{\Sigma_T} = 0 \end{cases}$

we get $\sup_{\tau \in [0,T]} \int_{\Sigma_\tau} |\phi|^2 \leq C \cdot \|\Box_g \phi\|^2_{L^2(R)}$

$$\Rightarrow \|\phi\|^2_{L^2(R)} \leq \int_0^T \int_{\Sigma_\tau} \phi^2 \cdot \frac{1}{|\nabla \tau|} d\mathrm{vol}_{\Sigma_\tau} \, d\tau \lesssim \int_0^T \int_{\Sigma_\tau} \phi^2 \lesssim \|\Box_g \phi\|^2_{L^2(R)}$$

So $|\mathcal{L}[\Box_g \phi]| \leq \|F\|_{L^2(R)} \|\phi\|_{L^2(R)} \lesssim \|F\|_{L^2(R)} \|\Box_g \phi\|_{L^2(R)}$

in other words $\mathcal{L}: Y \to \mathbb{R}$ is bounded (continuous) with the $L^2(R)$ topology. By Hahn-Banach: $\exists$ extension $\tilde{\mathcal{L}}: L^2(R) \to \mathbb{R}$ with the same bound.

By the Riesz representation theorem: Any continuous functional $\tilde{\mathcal{L}}: L^2(R) \to \mathbb{R}$ can be written as $\tilde{\mathcal{L}}[h] = \langle h, \psi \rangle$.

So, $\forall \phi \in X$: $\int_R F \cdot \phi = \mathcal{L}[\Box_g \phi] = \int_R \psi \cdot \Box_g \phi$, so $\psi$ solves ②.

Note: One can show that $\psi \in H^k(R)$ by applying the above argument for $\|\Box_g \phi\|_{H^{-k}}$ (i.e. using negative Sobolev exponent energy). This way, if $k \gg 1$, one can infer that $\psi$ actually solves ①.

Alternative proof: (see the book of Taylor)

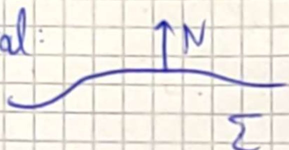
We will use the following general existence result in the real analytic category:

## Cauchy-Kovaleskaya theorem:

The initial value problem for a linear PDE with (real) analytic coefficients, analytic initial data posed on an analytic hypersurface which is $\boxed{\text{not characteristic}}$ has an analytic solution. The radius of analyticity of the solution can be bounded from below by a constant depending only on the radii of analyticity of the data, coefficients etc.

- For the wave equation: Non characteristic surfaces: Spacelike or timelike
- For elliptic equations: Every hypersurface is not-characteristic

In general:



is not characteristic for a $2^{nd}$ order PDE if $N^2(\psi)\big|_\Sigma$ is computable in terms of $\psi|_\Sigma$, $N(\psi)|_\Sigma$ and the coefficients of the eqn.

Remark: The above is a "deceivingly" general theorem:

For instance, for $\Delta\psi = 0$, the initial value problem with data on $\{x=0\}$ has no solution if the data are not real analytic.

So for
$$\begin{cases} \Box_g \psi = F \\ \psi\big|_{\Sigma_0} = 0, \quad \hat{n}(\psi)\big|_{\Sigma_0} = 0 \end{cases} \qquad (1)$$

Let $(g_n, F_n)$ be a sequence converging to $(g, F)$ in $C^k$ (for some $k \gg 1$) so that, for each $n$, $(g_n, F_n)$ have infinite radius of analyticity

(e.g. $F_n(x) = F * n^d \cdot e^{-n\frac{|x|^2}{2}}$, $d = \dim M$)

convolution

The Cauchy-Kovaleskaya theorem provides a sequence $\psi_n$ of solutions to $\begin{cases} \Box_{g_n} \psi_n = F_n \\ \psi_n|_{\Sigma_0} = 0, \quad \hat{n}(\psi_n)\big|_{\Sigma_0} = 0 \end{cases}$ with uniform radius of analyticity

Assume w.l.o.g that the whole domain R is contained inside the ball of analyticity.

We will show that there exists a subsequence s.t $\psi_n \xrightarrow{n\to\infty} \psi$ in $C^2$, so $\psi$ solves ①.

Higher order energy estimates for the sequence of wave equations:

$$\|\psi_n\|_{L^\infty_t H^k(\Sigma_t)} \leq C \|F_n\|_{H^{k-1}(R)} \underset{F_n \xrightarrow{C^\infty} F}{\xrightarrow{n\to\infty}} C\cdot\|F\|_{H^{k-1}(R)}$$

In fact: We control all k-th order derivatives in $L^2$, even $\hat{n}$-ones

Uniform in $n$, since it depends only on $\partial^k g_n$

So: The sequence $\psi_n$ is uniformly bounded in $L^\infty_t H^k(\Sigma_t)$.

<u>Rellich-Kondrachov theorem</u>: $H^{m_1}(\Omega) \hookrightarrow H^{m_2}(\Omega)$ is a compact embedding when $m_2 < m_1$ and $\bar{\Omega}$ is compact.

So $\psi_n$ is a precompact sequence in $L^\infty_t H^{k-1}(\Sigma_t)$

$\Rightarrow \psi_n \xrightarrow{L^\infty_t H^{k-1}_x} \psi$ along a subsequence.

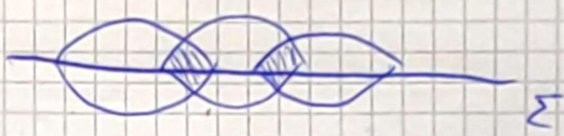
By the Sobolev embedding theorem (which we will prove shortly):

$\|f\|_{C^l(R^d)} \lesssim \|f\|_{H^m(R^d)}$ if $m > l + \frac{d}{2}$

So $\psi_n \longrightarrow \psi$ in $L^\infty_t C^2_x$, so $\psi$ solves ①. ▢

Corollary: Let $(M,g)$ be globally hyperbolic with Cauchy hypersurface $\Sigma$. Then
$$\begin{cases} \Box_g \psi = F \\ \psi|_\Sigma = \psi_0, \quad \hat{n}(\psi)|_\Sigma = \psi_1 \end{cases}$$
is globally well-posed. Moreover, the domain of dependence property holds.

Sketch of the proof: Inductively, by gluing the solutions in smaller regions, using uniqueness on the overlaps.




The Sobolev embedding estimate:

Def: Let $f: \mathbb{R}^n \to \mathbb{R}$.
$$\|f\|_{H^k(\mathbb{R}^n)} := \left( \sum_{|a|=0}^{k} \int_{\mathbb{R}^n} |\partial^a f|^2 \, dx \right)^{1/2},$$
where $\partial^a f = \partial_1^{a_1} \cdots \partial_n^{a_n} f$, $|a| = a_1 + \dots + a_n$

and $$\|f\|_{\dot{H}^k(\mathbb{R}^n)} := \left( \sum_{|a|=k} \int_{\mathbb{R}^n} |\partial^a f|^2 \, dx \right)^{1/2}$$ (homogeneous Sobolev semi-norm)

Sobolev embedding estimate: If $f$ is smooth and compactly supported:
$$\|f\|_{L^\infty} \leq C \, \|f\|_{H^k} \quad \text{if } k > \frac{n}{2}.$$
$C$ independent of $f$!

Remark: So it holds for any function in $H^k(\mathbb{R}^n)$, due to the density of smooth functions with compact support.

Proof: Fourier transform:

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\xi} f(x)\, dx$$

Inverse: $$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} \hat{f}(\xi)\, d\xi$$

So: $\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1}$ , $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$

and $\widehat{\partial^a f} = (-i\xi)^a \cdot \hat{f}(\xi)$ , $\xi^a = \xi_1^{a_1} \cdots \xi_n^{a_n}$

So $$\|f\|^2_{H^k} = \sum_{|a|\leq k} \|\partial^a f\|^2_{L^2} = \sum_{|a|\leq k} |\xi|^{2|a|} \|\hat{f}\|^2_{L^2} \sim \|(1+|\xi|)^k \hat{f}\|^2_{L^2}$$

Hence: $$\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1} = \int_{\mathbb{R}^n} |\hat{f}(\xi)|\, d\xi = \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|)^k} \cdot (1+|\xi|)^k |\hat{f}(\xi)|\, d\xi$$

$$\leq \underbrace{\left(\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|)^{2k}} d\xi\right)^{1/2}}_{<\infty \text{ if } 2k>n} \underbrace{\left(\int_{\mathbb{R}^n} (1+|\xi|)^{2k} |\hat{f}(\xi)|^2 d\xi\right)^{1/2}}_{\sim \|f\|_{H^k}}$$

$\square$

## Non-linear wave equations

General quasilinear wave equation:

$$\square_{g(\psi, \partial\psi)} \psi = N(\psi, \partial\psi)$$

↳ In local coordinates: $g_{\alpha\beta}(\psi, \partial\psi)$ is Lorentzian.

We want to study the initial value problem for initial data posed on a hypersurface $\Sigma$ which is spacelike with respect to $g(\psi, \partial\psi)|_\Sigma$ ← determined by the initial data, i.e. $\psi|_\Sigma$, $N(\psi)|_\Sigma$

We will consider the model example on $\mathbb{R}^{n+1}$:

$$\partial_a \left( G^{a\beta}(\psi)\, \partial_\beta \psi \right) = N(\psi, \partial\psi),$$

where $G^{a\beta}(0) = \eta^{a\beta}$, $|G^{a\beta} - \eta^{a\beta}| < \frac{1}{10 \cdot n}$

<u>Note</u>: $t=0$ is spacelike since $G^{00} < 0$, here $G^{a\beta}$ is the inverse of a Lorentzian metric.

(Recall that, for a general Lorentzian metric:

$$\Box_g f = \frac{1}{\sqrt{-\det g}}\, \partial_a \left( \sqrt{-\det g}\, g^{a\beta}\, \partial_\beta f \right)$$

<u>Local well-posedness theorem for quasilinear wave equations.</u>

<u>Thm</u>: Consider the initial value problem on $\mathbb{R}^{n+1}$:

$$\begin{cases} \partial_a \left( G^{a\beta}(\psi)\, \partial_\beta \psi \right) = N(\psi, \partial\psi) \\ \psi|_{t=0} = \psi_0, \quad \partial_t \psi|_{t=0} = \psi_1 \end{cases}$$

where $|G^{a\beta} - \eta^{a\beta}| < \frac{1}{10 \cdot n}$. Let also $k > 2n+2$

If $M = \|\psi_0\|^2_{H^k} + \|\psi_1\|^2_{H^{k-1}}$, then $\exists\, T = T(M) > 0$ and there <u>exists</u> a <u>unique solution</u> $\psi$ on $[0,T] \times \mathbb{R}^n$, such that:

$$\|\psi\|^2_{L^\infty_t H^k_x} + \|\partial_t \psi\|^2_{L^\infty_t H^{k-1}_x} \leq C \cdot M, \qquad C: \text{independent of } M, T, \psi.$$

Moreover, $\psi$ <u>depends continuously</u> on $(\psi_0, \psi_1)$.